THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2050 Mathematical Analysis (Spring 2018)
Tutorial on Mar 21

If you find any mistakes or typos, please email them to ypyang@math.cuhk.edu.hk

All through this note, A denotes the domain of a function and c is a cluster point of A.
δ-neighborhood of c : $V_\delta(c) = (c-\delta, c+\delta)$.
Deleted δ-neighborhood of c : $W_\delta(c) = (c-\delta, c+\delta) \setminus \{c\}$.

Part I: Comments

1. In the definition of limit of a function $f(x)$ at c:

$$\cdots \text{if } \ x \in A \ \text{ and } \ 0 < |x-c| < \delta, \text{ then} \cdots$$

you should also notice the words in red. To see what it means, let's consider the function $f(x) = \sqrt{x}$ defined on $A = [0, \infty)$ and $c = 0$. Then the function is not defined in any left-hand neighborhood of c.

However, from the definition we can check that $\lim\limits_{x\to 0} = \lim\limits_{x\to 0+} \sqrt{x} = 0$. Because we can choose $\delta(\varepsilon) = \varepsilon^2$ and then

$$x \in A \ \text{ and } \ 0 < |x-c| < \delta \ \text{ means } \ x \geq 0, 0 < |x| < \varepsilon^2 \Longrightarrow 0 < x < \varepsilon^2.$$

2. The **Sequential Criterion** is analogous to **Theorem 3.4.2** for the subsequences:

A sequence $X = (x_n)$ of real numbers converges to a real number x if and only if any subsequence $X' = (x_{n_k})$ of X also converges to x.

Part II: Reference exercises.

1. Suppose (x_n) is a sequence of positive real numbers and satisfies $x_n + \dfrac{4}{x_{n+1}^2} < 3$, $\forall n$. Show that (x_n) is convergent and find its limit.

2. Suppose the sequence (x_n) is defined by

$$x_1 = \frac{1}{2}, \quad x_{n+1} = x_n^2 + x_n, \quad n = 1, 2, \cdots.$$

Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+x_n} = 2$.

Proof: It can be shown that $\lim\limits_{n\to\infty} x_n = +\infty$. (Otherwise (x_n) is convergent since it's a strict increasing sequence. Suppose $\lim\limits_{n\to\infty} x_n = L$, then $L = L^2 + L \Longrightarrow L = 0$, contradiction)

Now $x_{n+1} = x_n^2 + x_n = x_n(x_n + 1)$ gives

$$\frac{1}{1+x_n} = \frac{x_n}{x_{n+1}} = \frac{x_n^2}{x_n x_{n+1}} = \frac{x_{n+1} - x_n}{x_n x_{n+1}} = \frac{1}{x_n} - \frac{1}{x_{n+1}}.$$

Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{1+x_n} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1+x_k} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{x_k} - \frac{1}{x_{k+1}}\right) = \lim_{n\to\infty} \left(\frac{1}{x_1} - \frac{1}{x_{n+1}}\right)$$
$$= \lim_{n\to\infty} \left(2 - \frac{1}{x_{n+1}}\right) = 2.$$

3. Determine whether the converse statement of the **Order-preservation property** is true: if $\lim_{x\to c} f(x) = a$, $\lim_{x\to c} g(x) = b$ and $a \le b$, then there exists $\delta > 0$ such that whenever $0 < |x-c| < \delta$ we have $f(x) \le g(x)$.

4. Determine whether the following statements are true or false.

   (a) If $\lim_{x\to c} f(x) = a$, $\lim_{x\to c} g(x) = b$ and there exists a deleted δ-neighborhood in which $f(x) < g(x)$, then $a < b$.

   (b) If $\sin(f(x))$ has a limit at $c = 0$, then $f(x)$ also has a limit at 0.

   (c) Suppose $f(x) : \mathbb{R} \to \mathbb{R}$. If $\lim_{n\to\infty} f\left(\frac{a}{n}\right) = 0$ for all $a \in \mathbb{R}$, then $\lim_{x\to 0} f(x) = 0$.

   (d) Suppose $\lim_{x\to c} f(x) = A > 0$, $\lim_{x\to c} g(x) = B$, then $\lim_{x\to c} f(x)^{g(x)} = A^B$.

   (e) If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x)$ does not exist, then $\lim_{x\to c}(f+g)(x)$ does not exist.

   (f) If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x)$ does not exist, then $\lim_{x\to c}(fg)(x)$ does not exist.

   (g) If neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c}(f+g)(x)$ does not exist.

   (h) If neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c}(fg)(x)$ does not exist.

5. Compute the following limits.

   (a) $\lim_{x\to 0} x \left[\frac{1}{x}\right]$

   (b) $\lim_{x\to +\infty} \left[\sqrt{(x+a)(x+b)} - x\right]$

   (c) $\lim_{x\to 1} \frac{x^m - 1}{x^n - 1}, \quad m, n \in \mathbb{N}$

   (d) $\lim_{n\to\infty} \sin\left(\pi\sqrt{n^2+1}\right)$

   (e) $\lim_{x\to +\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x}\right)$

6. Let $f, g : (a, +\infty) \to \mathbb{R}$ and $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$. Show that $\lim_{x\to+\infty} g(f(x)) = +\infty$.

   Proof: From $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} g(x) = +\infty$, we have that $\forall M > 0$, $\exists L > 0$, s.t. $x > L \Rightarrow g(x) > M$. And for this fixed L, there exists $K > 0$ such that $x > K \Rightarrow f(x) > L$ and consequently $g(f(x)) > M$.

   Therefore, $\lim_{x\to+\infty} g(f(x)) = +\infty$.

7. Suppose $\lim_{x\to+\infty} f(x) = L$, show that $\lim_{x\to+\infty} \frac{[xf(x)]}{x} = L$.

   Solution: Notice that $x(f) - 1 < [xf(x)] \leq xf(x)$ and then use Squeeze Theorem.

8. Prove that $\lim_{x\to 0} f(x)$ exists $\Longleftrightarrow$ $\lim_{x\to 0} f(x^3)$ exists. Does the same conclusion hold for $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} f(x^2)$?

9. Suppose $f(x)$ is monotone on $A = (a, b)$ and there is a sequence (x_n) in A such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} f(x_n) = L$. Show that $\lim_{x\to a+} f(x) = L$.

10. Suppose $f(x)$ is strictly increasing on $A = [a, b]$ and there is a sequence (x_n) in A such that $\lim_{n\to\infty} f(x_n) = f(a)$. Show that $\lim_{n\to\infty} x_n = a$.

    Solutions to Problem 9 and 10 will be provided later if necessary.

11. **(Generalizations of the Sequential Criterion)**

    (a) Show that if we replace the condition "for every sequence $(x_n)\cdots$" in the **Sequential Criterion** by "for every monotone sequence (x_n)" then the conclusion still holds.

    (b) What if we require in addition that "for every sequence (x_n) satisfying $|x_{n+1} - c| < |x_n - c|\cdots$, while other conditions unchanged?

    (c) We can weaken the condition as: function $f(x)$ has a limit at $c \in \mathbb{R}$ if and only if for every sequence (x_n) in $A \setminus \{c\}$ that converges to c, the sequence $(f(x_n))$ converges. (Notice that we are not assuming that $(f(x_n))$ converges to the same limit)

12. (Optional) From $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$, establish the following results.

    (a) $\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e$

    (b) $\lim_{x\to 0} \frac{\ln(1 + x)}{x} = 1$

    (c) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \quad (a > 0)$.